

THE MINIMAL RESOLUTION OF A COINTERVAL EDGE IDEAL IS MULTIPLICATIVE

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ABSTRACT. We show that the minimal resolution of the quotient of the polynomial algebra over a field by a cointerval edge ideal can be given the structure of a DG-algebra.

1. INTRODUCTION

To a simple graph G on the vertex set $[n] = \{1, \dots, n\}$, one can associate an ideal I_G in the polynomial algebra $S = k[x_1, \dots, x_n]$ over the field k , by letting I_G be generated by all monomials $x_i x_j$ such that ij is an edge in G ; this ideal is known as the *edge ideal* of G . In recent years, the study of edge ideals has enjoyed a great deal of popularity, and several authors have worked on relating the graph-theoretical properties of G to the algebraic properties of I_G .

In this paper, we study the minimal resolution of $R_G = S/I_G$ in the case when G is a cointerval graph, which is a graph that is the complement of an interval graph. The resolution can be obtained as a special case either of results by Chen [Che10], or by Dochtermann and Engström [DE12]. Chen constructs the minimal resolution of R_G for all complements of chordal graphs, and since every interval graph is chordal, cointerval ideals are covered. Dochtermann and Engström construct the minimal resolution of I_G for all cointerval d -hypergraphs; our cointerval graphs being the case of $d = 2$.

In section 2, we describe the resolution, in section 3 we use algebraic Morse theory to construct a contracting homotopy of the minimal resolution F_\bullet , and in section 4 we use this contracting homotopy to construct a map $\mu : F_\bullet \otimes_S F_\bullet \rightarrow F_\bullet$ which we show gives a commutative and associative multiplication on F_\bullet making it into a DG-algebra.

Not every cyclic module S/I has the property that its minimal resolution is multiplicative, see Avramov [Avr81] for results on homological obstructions to the existence of DG-algebra structures, as well as examples of ideals I such that S/I does not have a multiplicative minimal resolution. For a good survey of much of the early works on the existence and non-existence of multiplicative structures on resolutions, see Miller [Mil92].

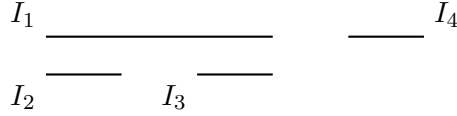
Nevertheless, several classes of resolutions of monomial ideals have been found to be multiplicative. Gameda [Gem76] and Fröberg [Frö79] have independently shown that the Taylor resolution of a monomial ideal is multiplicative; Peeva [Pee96] has shown that for I a stable monomial ideal, the

minimal resolution of S/I is multiplicative, and Sköldbberg [Skö11] has shown the corresponding result for matroidal ideals.

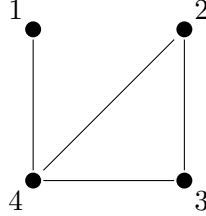
2. THE RESOLUTION AND ITS CONTRACTING HOMOTOPY

An *interval graph* is a graph whose vertices correspond to intervals of the real line, and where two vertices are adjacent if the corresponding intervals overlap. A *cointerval graph* is the complement of an interval graph.

Example 1. Consider the intervals $I_1 = [0, 3]$, $I_2 = [0, 1]$, $I_3 = [2, 3]$, $I_4 = [4, 5]$ as depicted below:



The corresponding cointerval graph G is thus



We will now describe the minimal resolution F_\bullet of R_G for G a cointerval graph. Dochtermann and Engström have constructed a polyhedral complex that supports the minimal resolution of a cointerval d -hypergraph; the resolution we will study is a special case of their construction. It is not hard to see that an interval graph is chordal, so the resolution F_\bullet is also a special case of Chen's construction of the minimal resolution of R_G for G such that its complement \bar{G} is chordal.

We will in the following assume that the vertex set is $[n]$ and that the vertices are ordered such that if the vertex i corresponds to the interval $[a_i, b_i]$, then $a_i \leq a_j$ whenever $i < j$.

For i a vertex of G , its *neighbourhood* $\text{nbhd}(i)$ is the set of all vertices j such that $ij \in E(G)$. Following Chen, we also define its *pre-neighbourhood* $\text{pnbhd}(i)$ to be all j in $\text{nbhd}(i)$ with $j < i$. We can then make the following observation.

Lemma 1. *Let i and j be vertices in G with $i < j$. Then $\text{pnbhd}(i) \subseteq \text{pnbhd}(j)$.*

Proof. If $i < j$ and $k \in \text{pnbhd}(i)$, it means that $[a_k, b_k] \cap [a_i, b_i] = \emptyset$, and thus $b_k < a_i \leq a_j$, so $k \in \text{pnbhd}(j)$. \square

The sets B_i which will consist of the basis elements of the resolution are now defined as follows: for the degree 0 part we let $B_0 = \{1\}$ and for the higher degrees we let B_d consist of the symbols $(\sigma|\tau)$ where $\sigma, \tau \subseteq [n]$ such that (1) σ and τ are disjoint and nonempty with $|\sigma \cup \tau| = d + 1$, (2) $\max \sigma < \min \tau$, and (3) $\{i, \min \tau\} \in E(G)$ for all $i \in \sigma$.

Now we can set $F_i = \bigoplus_{e \in B_i} S \cdot e$, and describe the differential in the complex F_\bullet :

$$F_\bullet : \quad 0 \longrightarrow F_r \xrightarrow{d_r} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} R_G \longrightarrow 0$$

by

$$\begin{aligned} d(i|j) &= x_i x_j \\ d(\sigma|\tau) &= \sum_{i \in \sigma} (-1)^{\alpha_1(\sigma, \tau, i)} x_i (\sigma \setminus i|\tau) \\ &\quad + \sum_{i \in \tau} (-1)^{\alpha_2(\sigma, \tau, i)} x_i (\sigma|\tau \setminus i) \end{aligned}$$

where

$$\alpha_1(\sigma, \tau, i) = |\tau| + |\{j \in \sigma \mid j > i\}| \quad \alpha_2(\sigma, \tau, i) = |\{j \in \tau \mid j > i\}|,$$

and where we interpret non-existent basis elements occuring in the formula as zero. By setting $\deg_{\mathbb{N}^n}(\sigma|\tau) = \deg_{\mathbb{N}^n} \left(\prod_{i \in (\sigma \cup \tau)} x_i \right)$ we get a complex of \mathbb{N}^n -graded modules, since it is clear that the differential respects this grading.

Theorem 1 (Chen, Dochtermann–Engström). *Given a cointerval graph G , the complex F_\bullet defined above is the minimal free \mathbb{N}^n -graded resolution of R_G .*

Proof. It is easy to see that the complex F_\bullet is the chain complex of the polyhedral complex that Dochtermann and Engström describe in [DE12], for the special case of an edge ideal of a cointerval (non-hyper)-graph.

Alternatively, the definition of F_\bullet can be seen to agree with Chen's resolution, [Che10, Construction 3.4] by virtue of the conclusion of Lemma 1 and the last remark in Chen's construction. \square

3. A CONTRACTING HOMOTOPY

In this section we will use methods of algebraic Morse theory to define a contracting homotopy on the resolution. The notation we will use is the same as in [Skö11], whither the reader is referred for reference.

In order to construct the contracting homotopy on F_\bullet , we consider F_\bullet to be a based complex of k -vector spaces with basis elements $x^\alpha(\sigma|\tau)$, and we will construct a Morse matching M on the directed graph Γ_{F_\bullet} . To help us show that the matching we are about to define is a Morse matching, we partially order the elements of B_d by letting $(\sigma_1|\tau_1) \prec (\sigma_2|\tau_2)$ if (1) $\max \tau_1 > \max \tau_2$, or (2) $\max \tau_1 = \max \tau_2$, and $\min \sigma_1 < \min \sigma_2$.

We define three sets of edges of Γ_{F_\bullet} : M_1 , M_2 and M_3 , the union of which will be our partial matching.

First, we let M_1 consist of the edges

$$x^\alpha(\sigma|\tau \cup j) \rightarrow x^\alpha x_j(\sigma|\tau), \quad j \geq \max(\text{supp } \alpha), j > \max(\tau).$$

There are now two types of unmatched vertices; first we have the vertices x^α , and then the vertices $x^\alpha(\sigma|\tau)$ where $|\tau| = 1$ and $\max(\text{supp } \alpha) \leq \max \tau$.

Next, we let M_2 be the edges

$$x^\alpha(i \cup \sigma|j) \rightarrow x^\alpha x_i(\sigma|j)$$

satisfying

$$i \in \text{nbhd}(j), \quad i < \min \sigma, \quad i \leq \min(\text{supp } \alpha \cap \text{nbhd}(j))$$

in the induced subgraph on the vertices M_1^0 . The vertices in $(M_1 \cup M_2)^0$ are then all x^α and the $x^\alpha(i|j)$ for which $j \geq \max \text{supp } \alpha$, and $i \leq \min(\text{supp } \alpha \cap \text{nbhd}(j))$, so we let M_3 be the set of edges

$$x^\alpha(i|j) \rightarrow x^\alpha x_i x_j, \quad (i|j) \prec\text{-minimal such that } x_i x_j | x^\alpha x_i x_j.$$

And, finally, we set $M = M_1 \cup M_2 \cup M_3$, and we get the unmatched vertices $M^0 = \{x^\alpha \mid x^\alpha \notin I_G\}$.

Lemma 2. *The set M is a Morse matching on Γ_{F_\bullet} .*

Proof. It is clear from construction that M is a partial matching, so we need to show that there are no infinite paths in $\Gamma_{F_\bullet}^M$. We can see that if we have an elementary reduction path from $x^\alpha(\sigma|\tau)$ to $x^\beta(\sigma'|\tau')$, then $(\sigma'|\tau') \prec (\sigma|\tau)$ which shows that the length of the directed path between vertices in the same degree is bounded. \square

Since M is a Morse matching with critical vertices M^0 concentrated in degree 0, we get a contracting homotopy φ as in [Skö06, Lemma 2], which can be described in terms of reduction paths, see Jöllenbeck and Welker [JW09] and Sköldbberg [Skö11]. We will next define a k -linear map c ; and then show that c coincides with the contracting homotopy φ .

We will need to distinguish between three types of basis elements in order to describe c :

- (1) x^α .
- (2) $x^\alpha(\sigma|\tau)$ where $|\tau| = 1$.
- (3) $x^\alpha(\sigma|\tau)$ where $|\tau| \geq 2$.

To the basis element $x^\alpha(\sigma|\tau)$, we associate sets C_1 , C_2 and C_3 by

$$\begin{aligned} C_1 &= \{i \mid i \in \text{supp } \alpha, i > \max \tau\} \\ C_2 &= \{i \mid i \in \text{supp } \alpha, i < \min \sigma, \{i, \min \tau\} \in E(G)\} \\ C_3 &= \{i \mid i \in \text{supp } \alpha, i < \min \sigma, i < \max \text{supp } \alpha, \{i, \max \text{supp } \alpha\} \in E(G)\} \end{aligned}$$

and in case the corresponding set is non-empty, we let $m_1 = \max C_1$, $m_2 = \min C_2$ and $m_3 = \min C_3$.

For the basis elements x^α we now let

$$c(x^\alpha) = \begin{cases} \frac{x^\alpha}{x_i x_j}(i|j), & \text{if } x^\alpha \in I_G, (i|j) \prec\text{-minimal such that } x_i x_j | x^\alpha, \\ 0, & \text{otherwise.} \end{cases}$$

Turning to the basis elements $x^\alpha(\sigma|\tau)$ where $|\tau| = 1$, $\tau = \{i\}$ next, we set

$$c(x^\alpha(\sigma|\tau)) = \begin{cases} \frac{x^\alpha}{x_{m_1}}(\sigma|\tau \cup m_1) \\ \quad + (-1)^{|\sigma|+1} \frac{x^\alpha x_i}{x_{m_1} x_{m_3}}(m_3 \cup \sigma|m_1), & \text{if } C_1 \neq \emptyset, C_3 \neq \emptyset, \\ \frac{x^\alpha}{x_{m_1}}(\sigma|\tau \cup m_1), & \text{if } C_1 \neq \emptyset, C_3 = \emptyset, \\ (-1)^{|\sigma|+1} \frac{x^\alpha}{x_{m_2}}(m_2 \cup \sigma|\tau), & \text{if } C_1 = \emptyset, C_2 \neq \emptyset, \\ 0, & \text{if } C_1 = \emptyset, C_2 = \emptyset. \end{cases}$$

Lastly, we treat the basis elements $x^\alpha(\sigma|\tau)$ where $|\tau| \geq 2$ and let

$$c(x^\alpha(\sigma|\tau)) = \begin{cases} \frac{x^\alpha}{x_{m_1}}(\sigma|\tau \cup m_1), & \text{if } C_1 \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 3. *The map c is an \mathbb{N}^n -graded contracting homotopy of F_\bullet such that $c^2 = 0$ and $c(e) = 0$ for all $e \in \bigcup_i B_i$.*

Proof. Let φ be the homotopy we get from the Morse matching M ; we shall see that $c = \varphi$.

First we look at the basis element $v = x^\alpha$. We have two cases, if $x^\alpha \in I_G$, then x^α is matched with $v' = x^\alpha/x_i x_j \cdot (i|j)$ where $(i|j)$ is minimal with respect to \prec . There are no elementary reduction paths originating in v' , so we can conclude that in this case $c(v) = v' = \varphi(v)$. In the case $x^\alpha \notin I_G$, we have that $x^\alpha \in M^0$, so $c(v) = 0 = \varphi(v)$.

Next, we turn to elements $v = x^\alpha(\sigma|j)$. If $C_1 \neq \emptyset$, $v \in M^-$ and is matched with $v' = x^\alpha/x_{m_1} \cdot (\sigma|jm_1)$. There is an elementary reduction path from v' to $v'' = x^\alpha x_j(m_3 \cup \sigma|m_1)$ precisely when $C_3 \neq \emptyset$. It is easy to see that there are no elementary reduction paths starting in v'' , so after verifying the signs, we can see that $c(v) = \varphi(v)$ when $C_1 \neq \emptyset$. If $C_1 = \emptyset$, we have that $x^\alpha(\sigma|j) \in M^-$ precisely when $C_2 \neq \emptyset$, in which case v is matched with $v' = x^\alpha/x_{m_2} \cdot (m_2 \cup \sigma|j)$ and there are no elementary reduction paths from v' , so $c(v) = \varphi(v)$ in this case as well.

Lastly, we look at the elements $v = x^\alpha(\sigma|\tau)$ where $|\tau| \geq 2$. Here we can see that $v \in M^-$ precisely when $C_1 \neq \emptyset$, in which case v is matched with $v' = x^\alpha/x_{m_1} \cdot (\sigma|\tau \cup m_1)$. There are no elementary reduction paths from v' which shows that $c(v) = \varphi(v)$ for these elements too.

It is clear from the definition that c respects the multidegree, and since $c(v) = 0$ for all elements in M^+ , we can see that $c^2 = 0$ and $c(e) = 0$ for all S -basis elements in B_m .

□

4. THE MULTIPLICATIVE STRUCTURE

Now we are in a position that allows us to define the multiplication making F_\bullet into a DGA. Just like in [Skö11], we are going to use the following result in the construction.

Lemma 4. *Suppose that X_\bullet and Y_\bullet are complexes of S -modules, where $X_n = S \otimes_k V_n$ and $Y_n = S \otimes_k W_n$ for k -spaces V_n and W_n , $n \geq 0$. Furthermore, suppose that Y_\bullet is acyclic, with a contracting homotopy c satisfying $c^2 = 0$. Then, every S -linear map $\varphi_0 : X_0 \rightarrow Y_0$ has a unique lifting to a chain map $\varphi : X_\bullet \rightarrow Y_\bullet$ satisfying $\varphi(V_n) \subseteq \text{Im } c$. This map is defined inductively by*

$$\varphi_{n+1}(\bar{x}) = c\varphi_n d(\bar{x}), \quad \bar{x} \in V_{n+1}.$$

Proof. This is a special case of [ML63, Theorem IX.6.2]. \square

We now let μ be the map $\mu : F_\bullet \otimes_S F_\bullet \rightarrow F_\bullet$ that is the lifting of the canonical isomorphism $\mu_0 : F_0 \otimes_S F_0 = S \otimes_S S \rightarrow S = F_0$ using the contracting homotopy c from the previous section. This will be our proposed product on F_\bullet so we will henceforth write $x \star y$ for $\mu(x \otimes y)$.

Lemma 5. *For all basis elements x, y of F_\bullet we have*

- (1) $d(x \star y) = d(x) \star y + (-1)^{|x|} x \star d(y)$.
- (2) $x \star y = (-1)^{|x||y|} y \star x$.
- (3) $1 \star x = x \star 1 = x$.

Proof. Claim (1) just expresses that μ is a chain map. For (2), we let $\tau : F_\bullet \otimes_S F_\bullet \rightarrow F_\bullet \otimes_S F_\bullet$ be defined on basis elements x, y by $\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x$. Now μ and $\mu \circ \tau$ are chain maps lifting the same map in degree 0 and both mapping basis elements to $\text{Im } c$; so by Lemma 4, they must be equal. Claim (3) is proven by induction on the degree of x . \square

Let us now define a map $\partial : F_n \rightarrow F_{n-1}$, $n \geq 1$, by

$$\begin{aligned} \partial(i|j) &= x_i x_j \\ \partial(\sigma|\tau) &= x_{\max(\tau)}(\sigma|\tau \setminus \max(\tau)) \\ &\quad - (-1)^{|\tau|+|\sigma|} x_{\min(\sigma)}(\sigma \setminus \min(\sigma)|\tau), \end{aligned}$$

again treating any non-existent basis elements occurring as zero.

Its usefulness comes from that we can replace the real differential d by ∂ when reasoning about the multiplication, as the following lemma shows.

Lemma 6. *For basis elements $(\sigma_1|\tau_1)$, $(\sigma_2|\tau_2)$ we have*

$$c(d((\sigma_1|\tau_1)) \star (\sigma_2|\tau_2)) = c(\partial((\sigma_1|\tau_1)) \star (\sigma_2|\tau_2))$$

Proof. Consider the difference

$$c(d((\sigma_1|\tau_1)) \star (\sigma_2|\tau_2)) - c(\partial((\sigma_1|\tau_1)) \star (\sigma_2|\tau_2)) = c((d((\sigma_1|\tau_1)) - \partial((\sigma_1|\tau_1))) \star (\sigma_2|\tau_2))$$

A term occurring in $d((\sigma_1|\tau_1)) - \partial((\sigma_1|\tau_1))$ is either of the form $x_i(\sigma_1 \setminus i|\tau_1)$ where $i > \min(\sigma)$ or $x_i(\sigma_1|\tau_1 \setminus i)$ where $i < \max(\tau_1)$.

Now assume that $x_k(\sigma_3|\tau_3)$ occurs in a product $(\sigma_1 \setminus i|\tau_1) \star (\sigma_2|\tau_2)$ or $(\sigma_1|\tau_1 \setminus i) \star (\sigma_2|\tau_2)$, and assume further that $c(x_i x_k(\sigma_3|\tau_3)) \neq 0$. This means that either (i), $i \geq k$ and $i > \max(\tau_3)$, which implies that $i = \max(\tau_1)$, or (ii), $|\tau_3| = 1$, and $i < \min(\sigma_3)$. Now, in case (ii), if $k < i$, we would have that $\tau_3 = \{t\}$ where $t = \max(\tau_1 \cup \tau_2)$, so by Lemma 1 it would be the case that $k \in \text{pnbhd}(t)$, but then $x_k(\sigma_3|\tau_3) \in M^-$ which contradicts that $x_k(\sigma_3|\tau_3) \in M^+$. Thus $i \leq k$, so $i = \min \sigma_1$. \square

We will now give an explicit description of the multiplication in the simplest non-trivial case.

Lemma 7. *Let $(s_1|t_1)$ and $(s_2|t_2)$ be basis elements of degree 1 in F_\bullet . Then*

$$(s_1|t_1) \star (s_2|t_2) = \begin{cases} x_{s_1}(s_2|t_2 t_1) + x_{t_2}(s_1 s_2|t_1) & t_1 > t_2, s_1 < s_2, \\ x_{s_1}(s_2|t_2 t_1) & t_1 > t_2, s_1 = s_2, \\ x_{s_1}(s_2|t_2 t_1) - x_{t_2}(s_2 s_1|t_1) & t_1 > t_2, s_1 > s_2, \\ x_{t_1}(s_1 s_2|t_2) & t_1 = t_2, s_1 < s_2, \\ 0 & t_1 = t_2, s_1 = s_2, \\ -x_{t_2}(s_2 s_1|t_1) & t_1 = t_2, s_1 > s_2, \\ x_{t_1}(s_1 s_2|t_2) - x_{s_2}(s_1|t_1 t_2) & t_1 < t_2, s_1 < s_2, \\ -x_{s_2}(s_1|t_1 t_2) & t_1 < t_2, s_1 = s_2, \\ -x_{s_2}(s_1|t_1 t_2) - x_{t_1}(s_2 s_1|t_2) & t_1 < t_2, s_1 > s_2. \end{cases}$$

Proof. By the definition of the product map

$$(s_1|t_1) \star (s_2|t_2) = c(x_{s_1} x_{t_1}(s_2|t_2)) - c(x_{s_2} x_{t_2}(s_1|t_1))$$

from which the statement follows by using the definition of c . \square

Lemma 8. *Let $(\sigma_1|\tau_1)$ and $(\sigma_2|\tau_2)$ be basis elements of F_\bullet . If $x_k(\sigma_3|\tau_3)$ occurs in $(\sigma_1|\tau_1) \star (\sigma_2|\tau_2)$, then $\sigma_3 \subseteq \sigma_1 \cup \sigma_2$ and $\tau_3 \subseteq \tau_1 \cup \tau_2$.*

Proof. We use induction over $d = \deg(\sigma_1|\tau_1) + \deg(\sigma_2|\tau_2)$. If $d = 2$, the claim follows from Lemma 7. If $d \geq 3$, we look at $c(\partial((\sigma_1|\tau_1)) \star (\sigma_2|\tau_2))$. In the case of $\deg(\sigma_1|\tau_1) = 1$, so $(\sigma_1|\tau_1) = (i|j)$, it is equal to $c(x_i x_j(\sigma_2|\tau_2))$. The only terms that could occur in $c(x_i x_j(\sigma_2|\tau_2))$ are $x_j(i \cup \sigma_2|\tau_2)$, $x_i(\sigma_2|\tau_2 \cup j)$ and $x_l(i \cup \sigma_2|\tau_2 \setminus l \cup j)$, all of which satisfy the statement of the lemma.

Next we turn to the case of $\deg(\sigma_1|\tau_1) \geq 2$, and, letting $s = \min \sigma_1$, $t = \max \tau_1$, we consider

$$c(\partial((\sigma_1|\tau_1)) \star (\sigma_2|\tau_2)) = c(x_t(\sigma_1|\tau_1 \setminus t) \star (\sigma_2|\tau_2)) \pm c(x_s(\sigma_1 \setminus s|\tau_1) \star (\sigma_2|\tau_2)).$$

First, suppose that $x_l(\sigma_4|\tau_4)$ occurs in $(\sigma_1|\tau_1 \setminus t) \star (\sigma_2|\tau_2)$, then the only terms that can occur in $c(x_t x_l(\sigma_4|\tau_4))$ are $v_1 = x_l(\sigma_4|\tau_4 \cup t)$ and $v_2 = x_m(l \cup \sigma_4|t)$. Note that if v_2 occurs, we must have $l = \min(\sigma_1 \cup \sigma_2 \cup \tau_1 \cup \tau_2)$, so $l \in \sigma_1 \cup \sigma_2$. Next suppose that $x_l(\sigma_4|\tau_4)$ occurs in $(\sigma_1 \setminus s|\tau_1) \star (\sigma_2|\tau_2)$, the only term that can occur in $c(x_s x_l(\sigma_4|\tau_4))$ is then $v_3 = x_l(s \cup \sigma_4|\tau_4)$. By induction, we have in both cases that $\sigma_4 \subseteq \sigma_1 \cup \sigma_2$ and $\tau_4 \subseteq \tau_1 \cup \tau_2$, so all of v_1 , v_2 and v_3 satisfy the conclusions of the lemma, and since the multiplication is graded commutative, the above argument also shows that all terms occurring in $c((\sigma_1|\tau_1) \star \partial((\sigma_2|\tau_2)))$ also satisfy the conclusion of the lemma, and thus, by invoking Lemma 6, we have shown that all terms occurring in $(\sigma_1|\tau_1) \star (\sigma_2|\tau_2)$ satisfy the conclusion of the lemma, and we are done. \square

Lemma 9. *For basis elements $(\sigma_1|\tau_1)$, $(\sigma_2|\tau_2)$ in F_\bullet , we have that if $x_k(\sigma_3|\tau_3)$ occurs in the product $(\sigma_1|\tau_1) \star (\sigma_2|\tau_2)$, then $\max \tau_3 = \max(\tau_1 \cup \tau_2)$ and $|\tau_3| \geq |\tau_1| + |\tau_2| - 1$.*

Proof. For the first claim we observe that if $\max \tau_3 \neq \max(\tau_1 \cup \tau_2)$, then $k = \max(\tau_1 \cup \tau_2)$, which would imply that $x_k(\sigma_3|\tau_3) \in M^-$, but since $x_k(\sigma_3|\tau_3) \in \text{Im } c$, we know that $x_k(\sigma_3|\tau_3) \in M^+$.

For the second claim, we observe that

$$|\sigma_3| + |\tau_3| = |\sigma_1| + |\tau_1| + |\sigma_2| + |\tau_2| - 1$$

so by Lemma 8

$$\begin{aligned} |\tau_3| &= (|\sigma_1| + |\sigma_2| - |\sigma_3|) + |\tau_1| + |\tau_2| - 1 \\ &\geq |\tau_1| + |\tau_2| - 1. \end{aligned}$$

□

Lemma 10. *Let $(\sigma_1|\tau_1)$, $(\sigma_2|\tau_2)$ and $(\sigma_3|\tau_3)$ be basis elements of F_\bullet , then $(\sigma_1|\tau_1) \star ((\sigma_2|\tau_2) \star (\sigma_3|\tau_3)) \in \text{Im } c$.*

Proof. Suppose $x_j(\sigma_4|\tau_4)$ occurs in $(\sigma_2|\tau_2) \star (\sigma_3|\tau_3)$ and furthermore that $x_k(\sigma_5|\tau_5)$ occurs in $(\sigma_1|\tau_1) \star (\sigma_4|\tau_4)$.

Suppose that $x_j x_k(\sigma_5|\tau_5) \notin \text{Im } c$. Then we must have that $c(x_j x_k(\sigma_5|\tau_5)) \neq 0$, which can only happen if $c(x_j(\sigma_5|\tau_5)) \neq 0$. Since $\max \tau_5 = \max(\tau_1 \cup \tau_4) = \max(\tau_1 \cup \tau_2 \cup \tau_3)$, we have that $j < \min \sigma_5$ and that $|\tau_5| = 1$, so by lemma 9 this means that $|\tau_i| = 1$ for $1 \leq i \leq 4$ and we can define m_i , $1 \leq i \leq 5$ by $\{m_i\} = \tau_i$.

It cannot be the case that $k < \min \sigma_5$, since that would imply that one of km_1 , km_2 , or km_3 is in $E(G)$, and thus, by Lemma 1 and Lemma 8, that $km_5 \in E(G)$ which would mean that $x_k(\sigma_5|\tau_5) \in M^-$. This means that $\min \sigma_5 \leq \min \sigma_4$.

Therefore we can conclude that $j < \min \sigma_5 \leq \min \sigma_4$, so we have that one of jm_2 and jm_3 is in $E(G)$, so $jm_4 \in E(G)$, and $x_j(\sigma_4|\tau_4) \in M^-$ which contradicts that $x_j(\sigma_4|\tau_4) \in \text{Im } c$ and thus is in M^+ . □

Theorem 2. *For a cointerval graph G , the minimal resolution F_\bullet of I_G is a DGA over S .*

Proof. Lemma 5 gives that the proposed multiplication has a unit, satisfies the Leibniz rule and is graded commutative. It thus remains to see associativity. To this end we look at the two chain maps

$$\mu \circ (\mu \otimes 1), \mu \circ (1 \otimes \mu) : F_\bullet \otimes_S F_\bullet \otimes_S F_\bullet \longrightarrow F_\bullet.$$

Since they agree in degree 0; Lemma 4 tells us that it is enough to show that the images of basis elements under both maps lie in $\text{Im } c$.

Let e_1 , e_2 and e_3 be basis elements of F_\bullet . If any of them is of degree zero, and thus equal to 1, it is by Lemma 5 obvious that $e_1 \star (e_2 \star e_3)$ and $(e_1 \star e_2) \star e_3$, lie in $\text{Im } c$, so let us assume that this is not the case. Then, by Lemma 10 we know that

$$e_1 \star (e_2 \star e_3) \in \text{Im } c.$$

and that

$$(e_1 \star e_2) \star e_3 = (-1)^{|e_3|(|e_1|+|e_2|)} e_3 \star (e_1 \star e_2) \in \text{Im } c.$$

□

We conclude by calculating the full DGA-structure on the resolution of the graph from Example 1.

Example 2. Continuing with our example, we have the following S -basis elements in the resolution:

Degree	Basis elements
1	$(1 4), (2 3), (2 4), (3 4)$
2	$(12 4), (13 4), (23 4), (2 34)$
3	$(123 4)$

We can now get the products of elements of degree 1 from Lemma 7. Since the product is graded commutative we have zeros on the diagonal, and elements below the diagonal are the negative of their transposes, so we do not include them in the table.

\star	$(1 4)$	$(2 3)$	$(2 4)$	$(3 4)$
$(1 4)$		$x_1(2 34) + x_3(12 4)$	$x_4(12 4)$	$x_4(13 4)$
$(2 3)$			$-x_2(2 34)$	$x_3(23 4) - x_3(2 34)$
$(2 4)$				$x_4(23 4)$
$(3 4)$				

Next we can compute the products of an element of degree 1 with an element of degree 2. From the \mathbb{N}^n -homogeneity of \star it follows that $(\sigma_1|\tau_1)\star(\sigma_2|\tau_2) = 0$ if $|(\sigma_1 \cup \tau_1) \cap (\sigma_2 \cup \tau_2)| \geq 2$, so we leave those entries blank in the table, and only include entries which need to be calculated.

\star	$(12 4)$	$(13 4)$	$(23 4)$	$(2 34)$
$(1 4)$			$-x_4(123 4)$	0
$(2 3)$	0	$x_3(123 4)$		
$(2 4)$		$x_4(123 4)$		
$(3 4)$	$-x_4(123 4)$			

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